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ON THE PLANE PROBLEM OF THE THEORY OF ELASTICITY FOR MULTIPLY-CONNECTED DOMAINS WITH CYCLIC SYMMETRY

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We consider the problem of determining the stresses in a thin, homogeneous disc, weakened by N like holes situated at the same distance from the center and acted upon by a constant normal load applied along its periphery. Such a cyclically symmetric problem was solved by Buivol in [1], who reduced the Sherman integral equation [2, 3] along the boundary L of the region in question, to an equation along the part of L designated by l and lying within the angle $\theta_0 \leqslant \theta \leqslant \theta_0 + \tau$, where $\tau = 2\pi / N$ and θ is the angular coordinate of the points of l in the polar coordinate system chosen in the plane of the annulus in the usual manner, and θ_0 is arbitrary.

Such an approach utilizes the symmetry of the problem when the resulting equations are solved numerically and, unlike other methods [4–6], it does not impose any restrictions on the size and distribution of the holes, while a suitable choice of the norm in the method of least squares ensures uniform convergence of the complex potentials $\varphi(z)$ and $\psi(z)$ and their derivatives right up to their boundaries. Unfortunately, the paper [1] contains an error. The transformation of the function $\omega(t)$ under a rotation by the angle τ is determined with the accuracy of only up to its principal term, i.e. up to the limiting value of the function holomorphic outside the region in question (see [7] for a representation of holomorphic functions in terms of the Cauchy integrals). Such a limiting value affects the form of $\psi(z)$ and hence the result. In the present paper this value is determined with help of the condition of transformation of $\psi(z)$ under rotation, used in [1].

It is proved that $\omega(t)$ belongs to some subspace $W_2^3(L, \tau)$ of the space $W_2^3(L)$, constructed by taking into account the symmetry of the problem. The application of the method of least squares in $W_2^3(L, \tau)$ leads to an economic computational scheme. We give numerical results for $N = 4$ in the case of different disk-geometries. The method of solution can be easily extended to the

case of an arbitrary static load which does not violate the symmetry of the problem.

We introduce the following notations: S_+ is the domain under consideration, S_- is the complement of S_+ to the entire complex plane, L is the boundary of the domain (related neither to S_+ not to S_-), L_0 is the outer boundary of the disk of radius R, L_k is the boundary and z_k is the center of the k-th hole of radius r $(k = 1, 2, \ldots, N)$, $\varphi(z)$, $\psi(z)$ are the Kolosov functions, holomorphic in S_+, and t, t_0 are points on L.

According to [2, 3], $\varphi(z)$ and $\psi(z)$ are sought in the form

$$\varphi(z) = \frac{1}{2\pi i}\int_L \frac{\omega(t)}{t-z}\,dt + U(z), \qquad U = \sum_{k=1}^{N} \frac{b_k}{z-z_k}, \tag{1}$$

$$\psi(z) = \frac{1}{2\pi i}\int_L \frac{\overline{\omega(t)}}{t-z}\,dt - \frac{1}{2\pi i}\int_L \frac{\bar{t}\omega'(t)}{t-z}\,dt + U(z) \tag{2}$$

$$b_k = i\int_{L_k} \{\omega(t)\,d\bar{t} - \overline{\omega(t)}\,dt\}, \qquad k = 1, 2, \ldots N$$

Here $\omega(t)$ is a sufficiently smooth complex function on L and b_k are real constants.

The boundary condition leads us to an integral equation relative to $\omega(t)$

$$\omega(t_0) + \frac{1}{2\pi i}\int_L \omega(t)\,d\ln\frac{t-t_0}{\bar{t}-\bar{t}_0} - \frac{1}{2\pi i}\int_L \overline{\omega(t)}\,d\frac{t-t_0}{\bar{t}-\bar{t}_0} + \tag{3}$$

$$\sum_{k=1}^{N}\left\{\frac{b_k}{t_0-z_k} + \frac{b_k}{\bar{t}_0-\bar{z}_k}\left(1-\frac{t_0}{\bar{t}_0-\bar{z}_k}\right)\right\} - C_k = f(t_0) \quad \text{on } L_k$$

$$C_k = -\int_{L_k} \omega(t)\,dS, \qquad dS = |dt|$$

Rotation through an angle τ does not affect the state of stress of the disk. Therefore, except for nonessential terms, we have

$$\varphi(ze^{i\tau}) = e^{i\tau}\varphi(z) \tag{4}$$

$$\psi(ze^{i\tau}) = e^{-i\tau}\psi(z) \tag{5}$$

We write the function $U(z)$ in the form

$$U(z) = \int_L \frac{h(t)}{t-z}\,dt, \qquad h(t) = \begin{cases} 0, & t \in L_0 \\ \dfrac{b_k}{t-z_k}, & t \in L_k \end{cases} \tag{6}$$

Substituting (1) into (4) and taking into account (6), we obtain

$$\omega(te^{i\tau}) = e^{i\tau}\omega(t) + H_0(t) + q(t)$$

$$q(t) = \begin{cases} 0, & t \in L_0 \\ \dfrac{b_k e^{i\tau} - b_{k+1}e^{-i\tau}}{t-z_k}, & t \in L_k, \quad k = 1, 2, \ldots, N \end{cases}$$

Here $H_0(t)$ is the boundary value of a function which decreases at infinity and is holomorphic in S_-, and will be determined later. In [1], the function $H_0(t)$ was arbitrarily assumed identically equal to zero.

In formula (6) as well as below, due to the periodicity condition, we assume $b_0 = b_N$, $b_{N+1} = b_1$ and we perform in (2) the change of variable $\xi = ze^{-i\tau}$. Making use of (5) and (6), we obtain

$$\frac{1}{2\pi i}\int_L \frac{H_0(t)}{t-\xi}\,dt + \frac{e^{-2i\tau}}{2\pi i}\int_L \frac{H_0(t)}{t-\xi}\,d\bar{t} - \frac{e^{-2i\tau}}{2\pi i}\int_L \frac{\bar{t}H_0(t)}{(t-\xi)^2}\,dt = \tag{7}$$

$$\sum_{k=1}^{N} \frac{b_k - b_{k+1}}{z - z_k} e^{-i\tau} - e^{-2i\tau}\sum_{k=1}^{N} \beta_k\left[\frac{\bar{z}_k}{(\xi - z_k)^2} + \frac{r^2}{(\xi - z_k)^3}\right]$$

$$\beta_k = b_k e^{i\tau} - b_{k+1} e^{-i\tau}$$

The integrals which contain $q(t)$ are computed with the aid of residues. With the aim of forming an equation of type (3) relative to $H(t) = e^{-i\tau} H_0(t)$, we first compute

$$d_k = i\int_{L_k} \{H(t)\,d\bar{t} - \overline{H(t)}\,dt\} = b_{k+1} - b_k \tag{8}$$

Noting that the function

$$p(\xi) = \int_L \frac{H(t)}{t-\xi}\,dt$$

is identically equal to zero in S_+, we add the expression $p(\xi) + \xi\overline{p'(\xi)}$ to the left-hand side of (7). Making ξ tend to t_0 and making use of the formula (8) and the relation $(t - z_k)(\overline{t} - \bar{z}_k) = r^2$, we arrive at the equation

$$H(t_0) + \frac{1}{2\pi i}\int_L H(t_0)\, d\ln\frac{t - t_0}{\bar{t} - \bar{t}_0} - \frac{1}{2\pi i}\int_L \overline{H(t)}\, d\frac{t - t_0}{\bar{t} - \bar{t}_0} +$$

$$\sum_{k=1}^{N} \frac{\bar{d}_k}{\bar{t}_0 - \bar{z}_k} = -\sum_{k=1}^{N} e^{i\tau}\beta_k\left[\frac{z_k}{(\bar{t}_0 - \bar{z}_k)^2} + \frac{r^2}{(\bar{t}_0 - \bar{z}_k)^3}\right]$$

whose solution is

$$H(t) = \begin{cases} 0, & t \in L_0 \\ d_k\left[\dfrac{1}{t - z_k} - \dfrac{t}{(\bar{t} - \bar{z}_k)^2}\right] - e^{i\tau}\beta_k \dfrac{t}{(\bar{t} - \bar{z}_k)^2}, & t \in L_k \end{cases} \tag{9}$$

Since $H(t)$ is the boundary value of a function which is holomorphic in S_-, we must have $b_{k+1} = b_k$. Reverting to $H_0(t) = H(t)\,e^{i\tau}$, we obtain finally

$$\omega(te^{i\tau}) = e^{i\tau}\omega(t) + \begin{cases} 0, & t \in L_0 \\ e^{2i\tau}\beta\left[\dfrac{1}{t - z_k} - \dfrac{t}{(\bar{t} - \bar{z}_k)^2}\right], & t \in L_k \end{cases} \tag{10}$$

here the subscript of β_k is omitted since all b_k are equal.

In order to solve (3) by the method of least squares, we choose a fundamental system of functions in $W_2^3(L)$

$$\left(\frac{t}{c_1}\right)^m, \quad \left(\frac{\bar{t}}{c_1}\right)^{m+1}, \quad c_1 < R \tag{11}$$

$$\left(\frac{c_2}{t - z_k}\right)^m, \quad \left(\frac{c_2}{(\bar{t} - \bar{z}_k)}\right)^{m+1}, \quad c_2 > r, \quad m = 0, 1, 2 \ldots$$

which is the direct union of the systems fundamental on L_p, $p = 0, 1, \ldots N$.

We form linear combinations of the elements of this system, satisfying the relation (10). These are, as one can easily see, combinations of the form

$$\sum_{p=0}^{n}\left\{f_p^{\circ}\left(\frac{t}{c_1}\right)^{pN+1}+g_p^{\circ}\left(\frac{\bar t}{c_1}\right)^{pN+N-1}\right\} \tag{12}$$

on L_0, and

$$\sum_{p=1}^{n}\left\{\frac{c_2^{\,p}f_p^{\,1}e^{i(p+1)k\tau}}{(t-z_k)^p}+\frac{c_2^{\,p}g_p^{\,1}e^{i(1-p)k\tau}}{(\bar t-\bar z_k)^p}\right\}-$$
$$8i\pi\,\mathrm{Re}\,(g_1^{\,1})\sin\tau\left[\frac{c_2e^{3i\tau}(1-e^{2ik\tau})}{(t-z_k)(1-e^{2i\tau})}-\right.$$
$$\left.\frac{c_2^2(1-e^{ik\tau})z_k}{(\bar t-\bar z_k)^2(1-e^{-i\tau})}-\frac{c_2^3e^{-i\tau}(1-e^{-2ik\tau})r^2}{(\bar t-\bar z_k)^3(1-e^{-2i\tau})}\right]$$

on L_k.

We take the closure of this linear set on the norm of W_2^3 and we denote the obtained space by $W_2^3(L,\tau)$. Let us prove that the desired function $\omega(t)$ belongs to this space. Since $\omega(t)$ belongs to $W_2^3(L)$, it follows that for sufficiently large n we have the inequality

$$\|\omega(t)-\nu(t)\|_{W_2^3}<\varepsilon,\qquad \nu=\sum_{i=1}^{n}\alpha_i y_i \tag{13}$$

where y_i are the elements of the fundamental system (11), α_i are constants, and ε is an arbitrary positive number. Performing in (13) a change of variable in the integral defining the norm $\xi=te^{-i\tau}$ and using (10), we obtain

$$\|\nu(te^{i\tau})-e^{i\tau}\nu(t)+\mu(t)\|_{W_2^3}\leqslant 2\varepsilon(1+\gamma)$$
$$\gamma=\sum_{k=1}^{N}\left\|\frac{1}{t-z_k}-\frac{t}{(\bar t-\bar z_k)^2}\right\|_{W_2^3},\qquad t\in L_k$$

The function $\mu(t)$ can be found for $\nu(t)$ from the right-hand side of (10). From here one can obtain the estimate $\|\omega(t)-\nu_1(t)\|_{W_2^3}\leqslant\varepsilon F$, where ν_1, belonging to $W_2^3(L,\tau)$ is constructed in terms of ν in the obvious manner and F is uniformly bounded.

We adduce the expressions for Ay_i computed with the aid of residues and the subsequent limiting process $z\to t$

$$A(t_1)^m=t^m+mt\bar t^{m-1}-mR^2\bar t^{m-2},\qquad m\geqslant 2$$
$$A(t_1)^m=2t^m,\qquad t_1\in L_0,\qquad m=0,1$$
$$A(\bar t_1)^m=\bar t^m,\qquad m\geqslant 1,\qquad t_1\in L_0$$
$$A\left(\frac{1}{t_2-z_k}\right)^m=\frac{1}{(t-z_k)^m}-\frac{mt}{(\bar t-\bar z_k)^{m+1}}+$$
$$\frac{m}{(\bar t-\bar z_k)^{m+1}}\left(z_k+\frac{r^2}{\bar t-\bar z_k}\right),\qquad m\geqslant 1$$
$$A\left(\frac{1}{\bar t_2-\bar z_k}\right)^m=\frac{1}{(\bar t-\bar z_k)^m},\qquad m\geqslant 2,\qquad t_2\in L_k$$
$$A\left(\frac{1}{\bar t_2-\bar z_k}\right)=-4\pi\left(\frac{1}{t-z_k}+\frac{1}{\bar t-\bar z_k}-\frac{t}{(\bar t-\bar z_k)^2}\right)$$

where A is the operator defined by the left-hand side of Sherman's equation. We have assumed, for simplicity, that $c_1=c_2=1$. To the expression for $A\,[1/(t_2-z_k)^m]$ the following term must be added:

$$2im\,(-1)^{m+1} m \sin\,(k\,(m+1)\,\tau)\; H^{-m-1}\left(\frac{1}{t} - \frac{1}{\bar{t}} + \frac{t}{\bar{t}^2}\right)$$

which in Eq. (3) corresponds to the term of the form

$$\frac{b_0}{t} + \frac{\bar{b}_0}{\bar{t}}\left(1 - \frac{t}{\bar{t}^2}\right)$$

$$b_0 = \frac{1}{2\pi i}\int_L \left\{\frac{\omega\,(t)}{t^2}\,dt + \frac{\overline{\omega\,(t)}}{\bar{t}^2}\,d\bar{t}\right\}, \qquad H = |\,z_k\,|$$

which is identically equal to zero.

The formulas for the coefficients $a_{lk} = (Ay_k,\ Ay_l)_{W_{23}}$ are constructed with the aid of the residues. We do not give them here because of their cumbersome nature.

For the numerical realization of the method, a program was set up and computations were carried out on the electronic computer ICL-1905E. The stresses $\sigma_\theta\,/\,P$ at the points of L and the radial displacements $u\,/\,u_0$ at the points of L_0 (P is the intensity of the load and u_0 is the displacement in a compact disk of the same radius) are shown in Fig. 1 for the case of holes which are close to each other, and in Fig. 2 for the case of holes

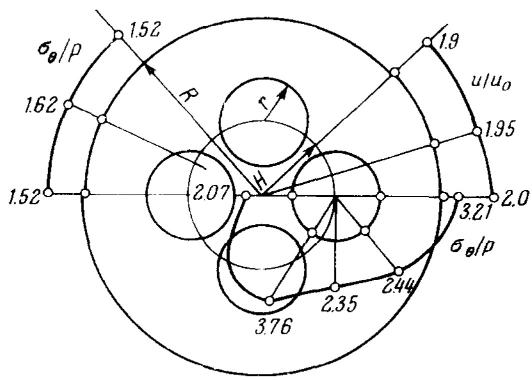

Fig. 1

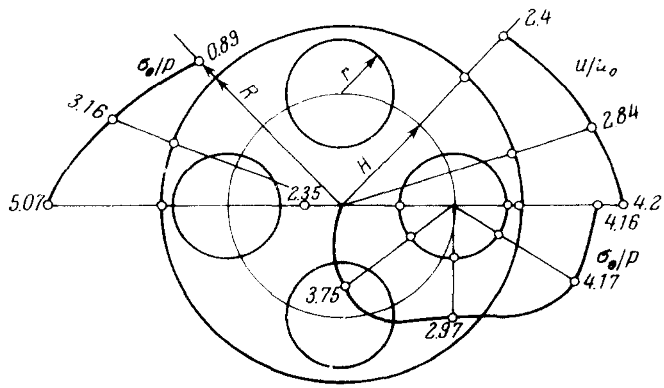

Fig. 2

which are close to the boundary of the disk. The relative dimensions of the disks were $r / R = 0.23$, $H / R = 0.4$ and $r / R = 0.3$, $H / R = 0.6$, respectively. The order of the normal system was chosen to be 22, the duration of the computations was about 30 minutes. The closeness of the obtained solution to the exact one was estimated from the relative error in the realization of the boundary conditions, being 4% in the first case and 2.5% in the second case. For $N = 6$, for the same disposition of the holes and the same accuracy, the duration of the computation increased to 42 minutes.

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ON A PROBLEM OF NONLINEAR DEFORMATION OF A CYLINDRICAL SHELL

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Problems of the axisymmetric deformation of elastic thin-walled shells of revolution, taking into account the finiteness of the displacements, have been examined sufficiently completely up to now for spherical type shells. Thus, numerical methods have been developed in [1-4] and solutions have been obtained for domes of diverse geometry under various external effects. It is shown below in the example of a long cylindrical shell that equilibrium modes of the rubber type of a flexible rod appear for shells of revolution whose Gaussian curvature is almost zero, under definite effects.

Let a cylindrical shell of thickness h and radius R (Fig. 1) be compressed uniformly by longitudinal stress resultants N and heated to the temperature $t(x) = {}^1/_2\ T \operatorname{sign}(x)$,